Mathematical Proof on Application of Pigeonhole Principle

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ABSTRACT

The pigeonhole principle that is applied in this problem has been aware of myself before the encounter of this problem, but I found it hard to apply the principle to a math problem. I have first experienced this problem in the process of the SUMaC appliance, where I couldn't solve the problem successfully, but I had in mind about the pigeonhole principle and I wanted to extend my understanding about the principle through this problem. In this problem, I will not only apply the principle, but I also will prove every and ultimately hope for an improvement in my math proving skills.

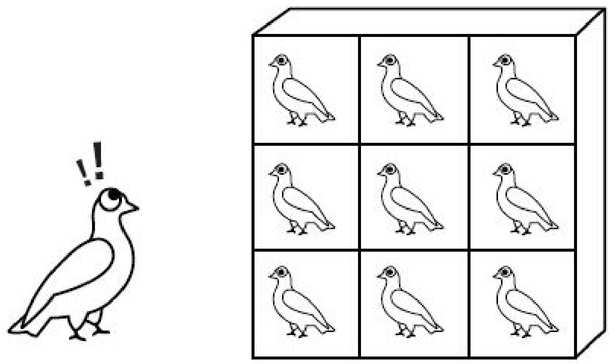
PROBLEM STATEMENT (2019 SUMAC ADMISSION EXAM PROBLEM 4)

An equilateral triangle has sides of length 1 cm.

1. Show that for any configuration of five points on this triangle (on the sides or in theinterior),thereisatleaston epairofthesefivepointssuchthatthedistance between the two points in the pair is less than or equal to .5cm.
2. Show that .5 (in part(a)) cannot be replaced by a smaller number even if there are 6 points.
3. IIf there are eight points, can .5 be replaced by a smaller number? Prove your answer.

PIGEONHOLEPRINCIPLE

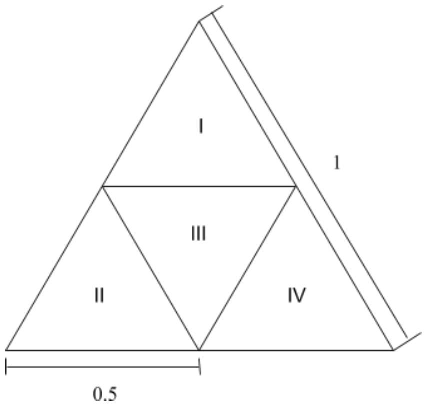
Suppose that a flock of n+1 pigeons flies into a set of n pigeonholes to roost. Because there are n+1 pigeons but only n pigeonholes, at least one of these n pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most n pigeons, one per hole, could be accommodated. This illustrates a general principle called the pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it. In the situation below, with ten pigeons and nine pigeonholes, there would be at least one pigeon left out, so one pigeonhole will have at least two pigeons in it if all ten pigeons are to be placed in a pigeonhole.



PART A):

Show that for any configuration of five points on this triangle (on the sides or in the interior), there is at least one pair of these five points such that the distance between the two points in the pair is less than or equal to .5 cm.

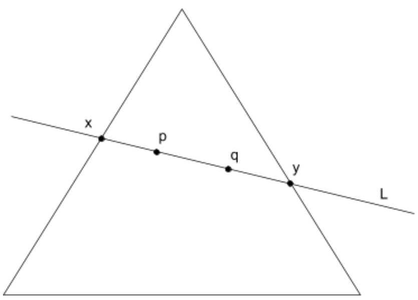
At least two points out of these five points must be within 0.5cm from each other when they are distributed within the triangle randomly. The five given points will be treated as five pigeons, thus I need at most four pigeonholes to be able to apply pigeonhole principle to the problem. I finally came up with the idea of how to partition the given triangle into four smaller regions that can be treated as pigeonholes (as shown in the <figure 1> below). According to the pigeonhole principle, one of the regions must have at least two points within the region because there are more points than the number of regions. Then all I need to prove is that the maximum distance between these two points in the same region is 0.5 cm to show that there is at least one pair of these five points such that the distance between the two points in the pair is less than or equal to 0.5cm.



<figure 1>

Lemma 1: The two points must be on the boundary of the triangle to have the maximum distance between them.

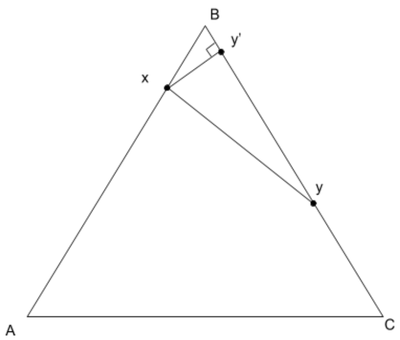
Proof of lemma 1: Without loss of generality, let's take a look at the region 1 as shown in <figure 1>. Any two points must be on the boundary of region I, not inside, in order to have maximum distance within the region. Let's choose two arbitrary points p and q in the region I as shown in the <figure 2> below. Then we draw a line L that is passing through points p and q. Let x and y be intersection points between line L and the boundary of the triangle.



<figure 2>

Then, it is clear that $|\overline{pq}| \leq |\overline{xy}|$, and the maximum distance between two points is attained when both points are on the boundary of the triangle.

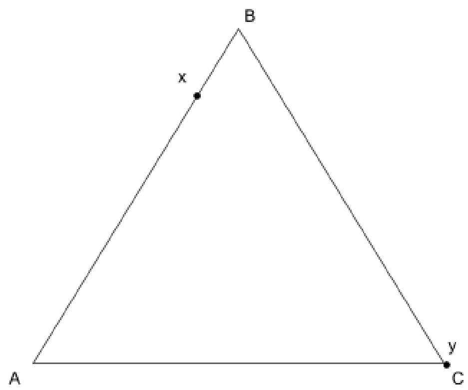
Lemma 2: The two points must be on the vertices of the smaller triangle in order to have the maximum distance between them.



<figure 3>

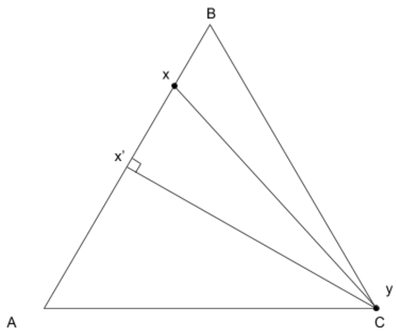
Proof of lemma 2: Let x be a point on the boundary AB and y be another point on the boundary BC as already proved in lemma 1. Let y' be the point of intersection between the boundary BC and the line that is passing through point x and BC perpendicularly.

In <figure 3>, if the point y shifts in between line segment BC, the distance between lines x and y $\sqrt{(\textit{distance between x and } y')^2 + (\textit{distance between y and } y')^2}$ will be and since the distance between x and y' is fixed, having more distance between y' and y will lengthen the distance between x and y, which will give the two points the furthest distance when y is in either point B or C.



<figure 4>

So far, we have proved that one of the two points must be on the vertex of the region. We now have to prove that the other point also must be on the vertex. Without loss of generality, let y be at vertex C.



<figure 5>

Let x' be the point of intersection between the boundary AB and the line that is passing through point y and AB perpendicularly.

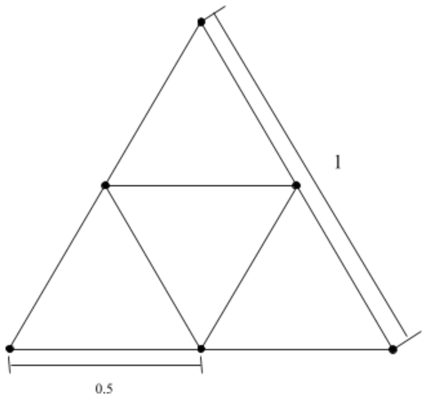
In <figure 5>, if point x shifts in between line segment AB, the distance between x and y will $\sqrt{(\textit{distance between x and } x')^2 + (\textit{distance between } x' \textit{ and y})^2}$ be. Since the distance between x' and y is fixed, having more distance between x and x' will lengthen the distance between x and y, which will give the two points the furthest distance when x is in either point A or B.

We have five pigeons (five points) and four pigeonholes (four smaller triangle regions), so there must be at least one smaller region that contains at least two points. Based on lemma 1 and lemma 2 that we already proved, these two points in the region must not only be on the boundary of the region but also on the vertices of the region. And the maximum distance between these two points is 0.5 where they are on the vertices. This implies that the maximum distance between any two points out of five points is 0.5.

PART B):

Show that .5 (in part(a)) cannot be replaced by a smaller number even if there are 6 points.

If the number is not going to be replaced with a smaller number, it is essential that the six points keep at least 0.5 units apart, and the following example shows that particular situation.



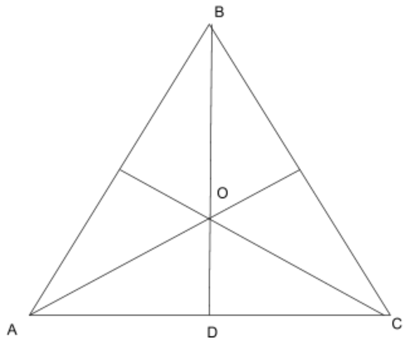
<figure 6>

With <figure 6> as a counterexample in a triangle of six points with exactly 0.5 units apart from each other, changing from five to six points will not replace the distance of .5 by a smaller number even if there are six points.

PART C):

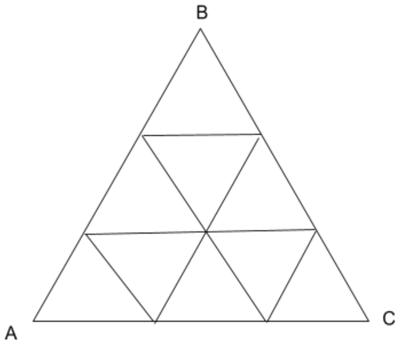
Part C): If there are eight points, can 0.5 be replaced by a smaller number? Prove your answer.

Based on the answers from part a and b, the triangle must be divided into five, six, or seven separate regions. The number of regions must not be more than seven because having eight or more subregions will cause the pigeonhole principle to be inapplicable. On the other hand, the reason that the number of subregions to be higher than 4 is because the maximum distance between two points within a subregion will have a higher chance of getting smaller than 0.5 if there are more subregions.



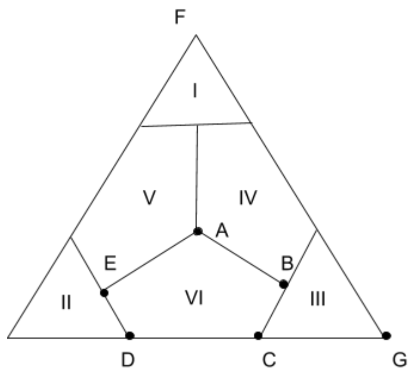
<figure 7>

In <figure 7> above, the big triangle is divided into six smaller subregions which are in appropriate intervals, but the maximum distance between a single subregion is greater than 0.5 (length of line segment AO is 0.577, since point o is the centroid of the triangle the length of AO is two-thirds of the altitude of the triangle.). Therefore, if the regions are divided as above, since line segment AO is longer than 0.5, 0.5 can not be replaced with a smaller number.



<figure 8>

<Figure 8> is another example that does not work for the question, this triangle, despite the fact that all of its lengths are less than 0.5, since the triangle is divided into nine regions, the pigeon hole principle is not plausible for this triangle. Since <figure 7 and 8> is not plausible for replacing 0.5 with a smaller number, we will be dividing the triangle as figure 9.



<figure 9>

In <Figure 9> above, Let x be the side length of the smaller triangular region I, II, or III (i.e. CG = 0.5). Then, the maximum distance between any two points is x, which is less than 0.5. Each side of the triangle is less than 0.5. We already proved that the maximum distance between any two points within the equilateral triangle region is less than or equal to the side length of the triangle.

The line segment with the longest distance must be less than 0.5, and the maximum lengths should be checked if it is less than 0.5. Also, x itself needs to be considered since it is also one of the segments. There would be ten pairs since it can only be on the edge as proven before by choosing two points out of 5 (5C2 = 10).

In <figure 9>, AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE are all of the line segments that need to be measured. Among these ten distinct line segments, AE, DE, CE, and AD can be disregarded because AE is equal to AB, DE is equal to BC, CE is equal to BD, and AD is equal to AC.

The line segments that certainly are not regarded as the longest can also be disregarded. AB is shorter than BE because in triangle ABE, ∠BAE is the largest angle with 120° implying that BE, the opposite side of the angle, is the longest segment. Also, BC and CD will not be taken into consideration because in triangle BCD, ∠BCD is the largest with 120° implying line segment BD, the opposite side of the angle, is the longest segment.

Overall, CG which is equal to x, AC, BD, and BE should be considered, and each of these line segments will be explained in terms of x.

$$\overline{AB}=\left(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}x}{2}\right)^2$$
$$\overline{BC}=\frac{x^2}{4}$$
$$\overline{AC}=\sqrt{\left(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}x}{2}\right)^2+\frac{x^2}{4}}$$
$$=\sqrt{\frac{3x^2-3x+1}{3}}$$

The length of AC can be computed using the Pythagorean Theorem as shown above.

$$\overline{AB}=\overline{AE}=\frac{\sqrt{3}}{3}-\frac{\sqrt{3}x}{2}$$
$$\overline{BE}=\sqrt{2\left(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}x}{2}\right)^2-2\left(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}x}{2}\right)^2(\cos 120^\circ)}$$
$$=1-\frac{3x}{2}$$

The length of BE can be computed using the Law of Cosines as shown above.

$$\overline{BC}=\frac{x}{2}$$
$$\overline{CD}=1-2x$$
$$\overline{BD}=\sqrt{\left(\frac{x}{2}\right)^2+(1-2x)^2-2\left(\frac{x}{2}\right)(1-2x)\left(\cos 120^\circ\right)}$$
$$=\frac{\sqrt{13x^2-14x+4}}{2}$$

The length of BD can be computed using the Law of Cosines as shown above.

Since not all of these values are linear, there would be different maximum values for different x values, which requires to view all ranges where each line segment is below 0.5 units. When all of the four line segments are less than 0.5 units, the overall condition will be satisfied.

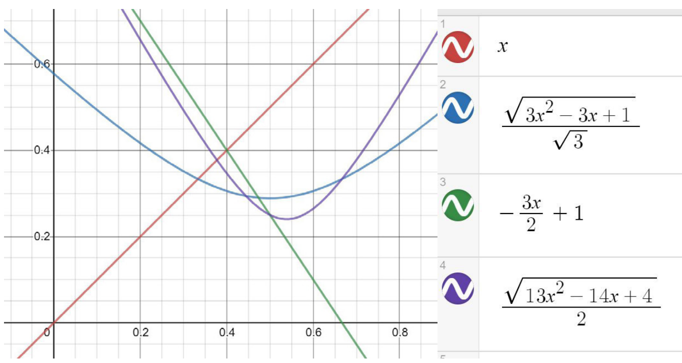
$$\overline{AC} < 0.5 \rightarrow 0.092 < x < 0.908$$
$$\overline{BE} < 0.5 \rightarrow x > 0.333$$
$$\overline{BD} < 0.5 \rightarrow 0.295 < x < 0.782$$
$$\overline{CG} < 0.5 \rightarrow x < 0.5$$

Ultimately, the possible domain of x is between 0.333 and 0.5. This is because in the domain between 0.333 and 0.5, the value of each line segment will all be less than 0.5. This domain for x proves that there is a possibility for a triangle to be divided with all lengths less than 0.5, therefore having eight points on a triangle will replace the distance between any two points by a smaller number than 0.5.

FURTHER INVESTIGATION:

It is already proven that 0.5 can be replaced. It is even possible to compute the exact value that can replace 0.5 graphically.

Relation Between x and The Length of Each Line Segment
(Created by Desmos)



<figure 10>

In <figure 10>, the lengths of segments CG (Red Line), AC (Blue Curve), BE (Green Curve), and BD (Purple Curve) are shown in relation to x. In order to replace 0.5 with the least possible value, it is required for the maximum value of four lengths at a given x to be as small as possible. When x is equal to 0.4, the maximum length of four line segments occurs at the intersection between green line (BE) and red line (CG), which is 0.4 as well, as shown in figure 10. To conclude, the smallest value that 0.5 can be replaced with is 0.4 when there are eight points given.