SOLUTIONS OF THE TOWER OF HANOI

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ABSTRACT

Tower of Hanoi is one of the most popular puzzles (see [1]-[2]). It was invented by the French mathematician Eduard Lucas in 1883. The tower consists of the eight discs, stringed on the one of the three peg s in order to reduce the size of discs. The problem is to move the entire tower to one of the other peg, bearing in each case only one disc,and without putting a larger disc over the smaller one.

Let us generalize the problem. Consider what would happen in the case of n discs. Let T_n is a minimum number of re-arrangements required to move n disks from one peg to another according to the rules of Lucas. Then $T_1 = 1$, $T_2 = 3$.

STATEMENT OF PROBLEM #1

Get the formulae of the T_n for arbitrary n.

Henry E. Dudeney proposed to solve this problem for a larger number of pegs. For the case of a 4-core tower there are many ways to achieve what we think the shortest number of moves, but still no way to characterize these solutions.

STATEMENT OF PROBLEM #2

Get the estimate of the number of rearrangements Q_n and generate the strategy of discs moving for 4-core tower.

SOLUTION

Experiments with three disks show that the crucial idea is to transfer two top disks from the first peg to another (intermediate) peg, according to the rule, provided for the transfer of two disks. Then it is necessary to transfer third disc onto a free peg and move two remaining disks over this one due to the same rule, mentioned above. This consideration gives the key of understanding the general rule of *n* disks moving:

1. Move *n-1* smaller disks on the any free peg (i.e. we have T_{n-1} rearrangements) ,`
2. Shift the largest disc (single shifting)
3. Moving the *n-1* smaller disks back to the largest disk (we have T_{n-1} rearrangements additionally)

   Thus, n disks can be moved by $2T_{n-1} + 1$ rearrangements, i.e. when $n > 0$,

$$T_n = 2T_{n-1} + 1.$$

Equality (1) is recursion. It allows us to compute T_n for any *n*.

Let's solve the recurrence relation. One of the methods consists of guessing the correct solution, followed by a proof that guess is correct. One can calculate $T_3 = 2 \quad 3+1=7$; $T_4=2 \quad 7+1=15$. Obviously $n > 0$ we obtain

$$T_n = 2^n - 1.$$

Let's prove (2) with help of mathematics induction.

1. Relation (2) is true for *n=1*
2. Let's suppose that (2) is true for *n-1*:

$$T_{n-1} = 2^{n-1} - 1.$$

We'll prove (2) with help of (3).

Actually, $T_n=2T_n-1+1=2 = 2 \quad (2^n-1 \; -1) +1= 2^n-2+1= 2^n-1$. Therefore, the original Lucas problem requires $2^8-1 = 255$ rearrangements.

Let's show the solution of the four-disc problem step by step.

We will enumerate disks by numbers and enumerate pegs by the Latin letters. We will move disks with numbers 1, 2 , 3, 4 between the cores **a**, **b**, **c** and **d** (initial position: disks are located on the peg ***a*** – 1st disk is green, 2nd disk is blue, 3rd disk is yellow, 4th disk is red - see Fig.1).

1st step: Move disc "1" on the peg **b**. (Fig.2)

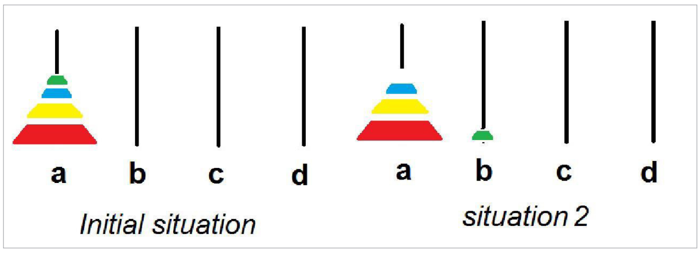

Figure 1 Figure 2

2nd step: Move disk “2” on the peg **c**. (Fig.3) 3rd step: Move disk “3” on the peg **d**. (Fig.4)

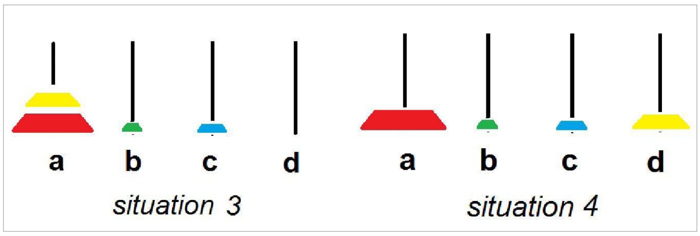

Figure 3 Figure 4

4th step: Move disc “1” on the peg c. (Fig.5) 5th step: Move disc “4” on the peg b. (Fig.6)

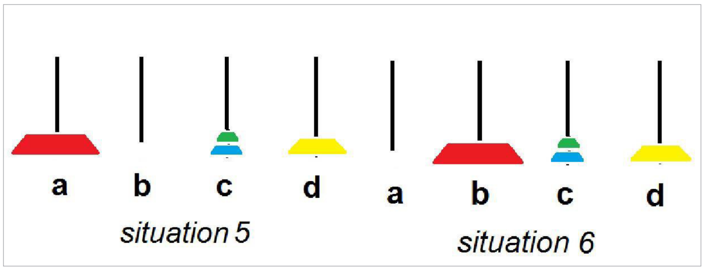

Figure 5 Figure 6

6th step: Move disk “3” on the peg **b.** (Fig.7) 7th step: Move disc “1” on the peg **a.** (Fig.8)

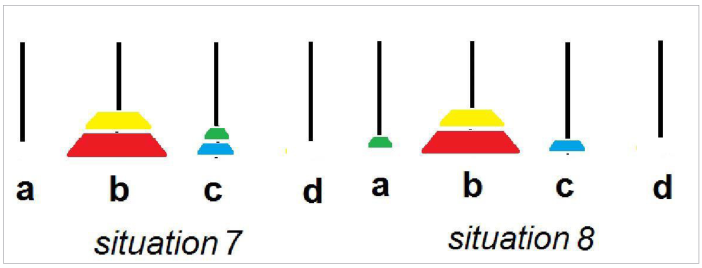

Figure 7 Figure 8

8th step: Move disk “2” on the peg **b.** (Fig.9) 9th step: Move disc “1” on the peg **b.** (Fig.10)

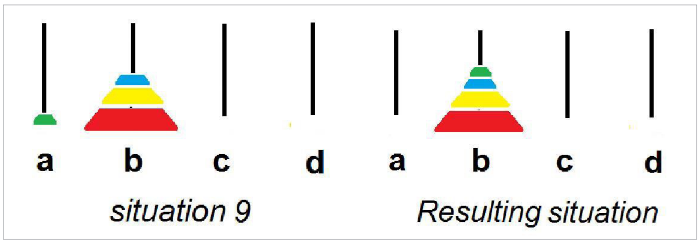

Figure 9 Figure 10

Solving this problem requires 9 steps , i.e .Q_4 = 9.

Step by step solution of the five-disk problem looks like this:

1st step: Move disc “1” on the peg **b**.
2nd step: Move disk “2” on the peg **c**.
3rd step: Move disk “3” on the peg **d**.
4th step: Move disk “2” on the peg **d**.
5th step: Move disc “4” on the peg **c**.
6th step: Move disc “1” on the peg **c**.
7th step: Move disk “5” on the peg **b**.
8th step: Move disc “1” on the peg **a**.
9th step: Move disc “4” on the peg **b**.
10th step: Move disk “2” on the peg **c**.
11th step: Move disk “3” on the peg **b**.
12th step: Move disk “2” on the peg **b**.
13th step: Move disc “1” on the peg **b**.

Solving this problem require 13 steps, i.e.Q_5 = 13.

One can show that Q_6 = 17, Q_7 = 25, Q_8 = 33, Q_9 = 41, Q_{10}=49.

These results allowed developing strategies for moving *n* disks. Let strung discs on a peg **a**.

1. 1) Move *n* - 4 upper disk using all four pegs and collect tower on one of them, such as **b** (Q_{n-4} movements).

1.2) Move the remaining four discs using three pegs **a**, **c** and **d** (peg **b**, which gathered upper discs, does not use), and collect them on one of pegs, such as **c**. (T_4 movements).

1.3) Move *n* - 4 discs from the peg **b** and collect them on the peg **c** using all four pegs. (Q_4 movements).

Thus, a recurrence relation allows us to estimate the minimum quantity of moves for 4-core Hanoi tower:

$Q_n \leq 2Q_{n-4}+15$	

The algorithm described above is similar to the *Frame–Stewart algorithm*, giving a *presumably* optimal solution for four pegs [3]. The last one can also be described recursively, but does not concern exact estimate of minimum number of discs rearrangements.

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