Mathematical Proof on Max/Min Problem

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PROBLEM STATEMENT (EXCERPT FROM 2019 STANFORD UNIVERSITY MATHEMATICS CAMP)

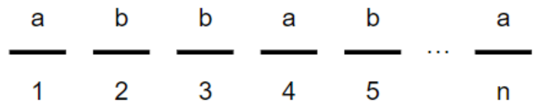
Suppose n is a positive integer. The (imaginary) sea of Babab has islands each of which has an n-letter name that uses only the letters “a” and “b,” and such that for each n-letter name that uses only the letters “a” and “b,” there is an island. For example, if n = 3, then Aaa, Aab, Aba, Baa, Abb, Bab, Bba and Bbb are the islands in the sea of Babab. The transportation system for Babab consists of ferries traveling back and forth between each pair of islands that differ in exactly one letter. For example, there is a ferry connecting Bab and Bbb since they differ only in the second letter.

1. How many islands and how many ferry routes are there in terms of n? Count the ferry route for both directions as a single ferry route, so for example, the ferry from Bab to Bbb is the same ferry route. Babab does not have much in the way of natural resources or farmland so nearly all food and supplies are provided by the Babab All Bulk Company (BABCO). The people of Babab (Bababians) desire easy access to a BABCO store, where “easy access” means there is a BABCO store on their own island or on one that they can get to with a single ferry ride. However, BABCO finds it uneconomical to give the people on one island easy access to two different BABCO stores, and BABCO is willing to deny some Bababians easy access to a BABCO store in order to meet this restriction.
2. In the cases n = 3, n = 4, and n = 5, what is the maximum number of stores that BABCO can build while satisfying the restriction than no one has easy access to more than one BABCO store? Be sure to prove your answer is optimal
3. Now suppose BABCO changes its strategy and decides it wants to be sure every Bababian has access to a BABCO store even if it means some Bababians have easy access to two stores. What is the minimum number of stores needed to satisfy this condition in the cases n = 3, n = 4, and n = 5?
4. Can you find optimal solutions to parts b and c for n = 6?

SOLUTION WITH PROOF

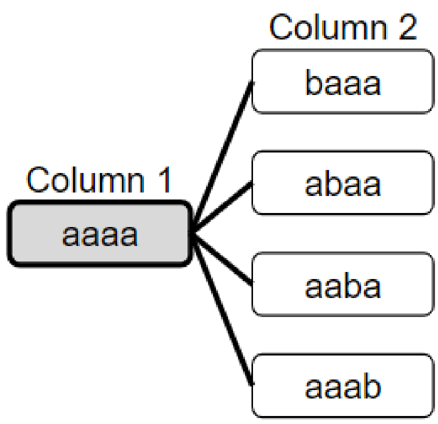
Calculation of Number of Island and Ferry Routes in terms of n

Each island has an n-letter name that consists of either a or b. Either of the two letters can occupy a space, which represents a letter from an n-letter name of an island.



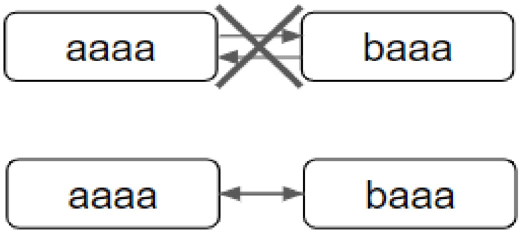
<figure 1>

There are n spaces, where each space has two options a or b as shown in the above given <*figure 1*>. The total number of n letter name islands is $2\times2\times2\times\ldots\times2= 2^n$.



<figure 2>

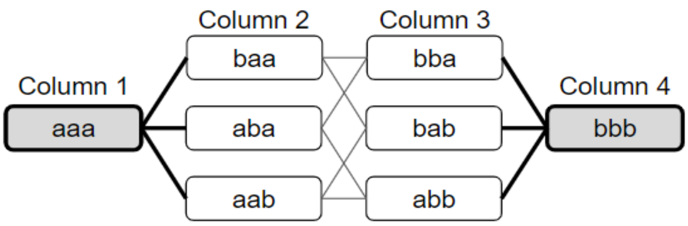
As shown in the <*figure 2*> in the previous page, each island has n routes connected to it. Since there are islands, the total number of routes that every island has is n multiplied by 2^n, or $n\times2^n$. However, since each route connects two islands in both directions, the total number of routes that every island has bidirectional connectivity between the routes as shown in the <*figure 3*> below. Therefore, it is necessary to divide the total number of routes that every island has by 2 in order to get the total number of routes without counting a route twice. As a result, the total number of routes equals $\frac{n\times2^n}{2}$.



<figure 3>

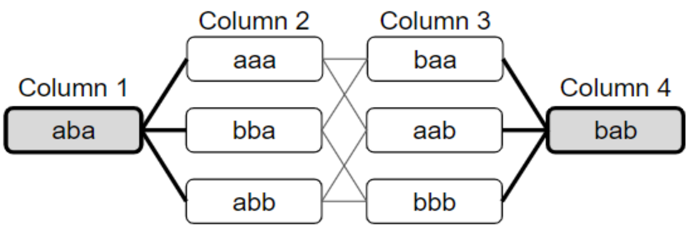
Maximum number of stores that BABCO can build

i. → n = 3



<figure 4>

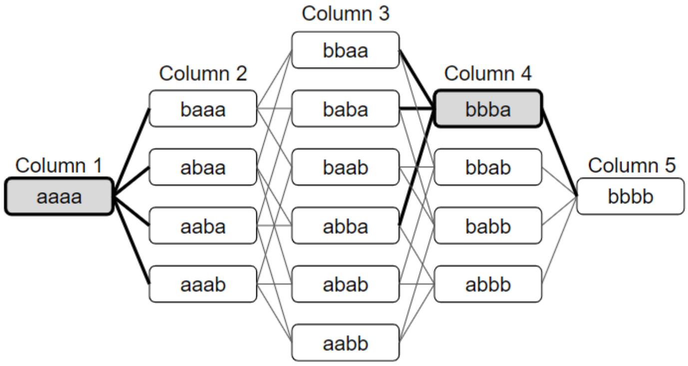
If there is a BABCO store on an island aaa in Column 1, then there cannot be any shop on islands in Column 2 in order to satisfy the restriction that no island can have more than one easy access to a BABCO store. In the <figure 4> above, the island aaa in Column 1 would have two easy accesses. This is against the restriction as one island is allowed to have one easy access if there were a store on one of the islands in Column 2. Additionally, islands in Column 3 cannot have any BABCO store on them, otherwise, the same would have resulted in providing more than one easy access to any island in Column 2. Since islands in Column 3 and 4 do not have any easy access, the island bab in Column 4 can have a store without violating the restriction. Therefore, the maximum number of BABCO stores for 3-letter name islands is 2.



<figure 5>

Even if an island other than the island aaa, such as the island aba shown in the <*figure* 5>, is required to have a store, the maximum number of BABCO stores for n = 3 is still 2. Therefore, the maximum number of BABCO stores for n = 3 is 2 without loss of generality.

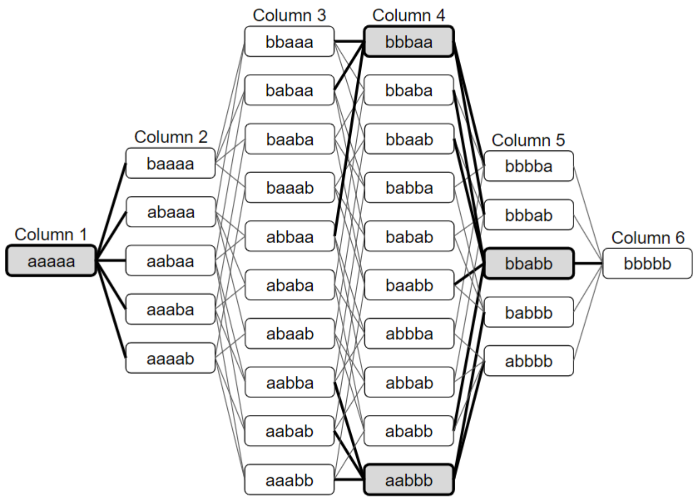
ii. → n = 4



<figure 6>

If there is a BABCO store on the island aaaa in Column 1 as shown in the <*figure* 6>, then there cannot be any store on islands in Column 2. Since the islands in Column 2 already have easy access, the islands in Column 3 cannot have any store. On the other hand, building a store on one of the islands in Column 4 would not give more than one easy access to any island. If a store is built on one of the islands in Column 4, then the island in Column 5 cannot have any store since the island in Column 5 already has easy access. Therefore, the maximum number of BABCO stores that can be built in the case of n = 4 without violating the easy access restriction is 2 without loss of generality.

iii. → n = 5



<figure 7>

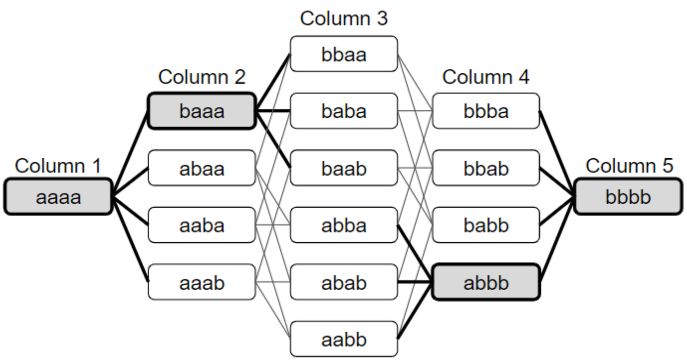
If there is a BABCO store on island aaaaa in Column 1 as shown in the <*figure* 7>, then there cannot be any BABCO store on the islands in Column 2 in order for the island aaaaa to have only one access. Since the islands in Column 2 already have easy access, there cannot be any BABCO store on the islands in Column 3. On the other hand, there can be BABCO stores on the islands in Column 4 since the islands in Column 3 and 5 do not have easy access. In order to maximize the number of BABCO stores on the islands in Column 4 without giving an island more than one easy access, BABCO stores can be built on the islands bbbaa and aabbb. These two stores do not give any island more than one easy access. The store on bbbaa gives easy access to the islands bbaaa, babaa, abbaa in column 3 and the islands bbbba and bbbab in column 4. The store on aabbb gives easy access to the islands aabba, aabab, and aaabb in Column 3 and the islands babbb and abbbb in Column 5. If there were additional stores in Column 4, then some of the islands in Column 5 would have more than one easy access. Therefore, there cannot be any more store in Column 4. Now we move on to Column 5. There is only one island bbabb in Column 5 that does not have easy access yet. If a store is built on the island bbabb, then it would give easy access to islands that do not have easy access yet. Now there cannot be more stores in both Column 5 and Column 6. The maximum number of stores for the n=5 case is 4.

Minimum number of stores needed to satisfy BABCO;s Change of Strategy

i. → n = 3

The maximum number of BABCO stores for islands in the case of n = 3 without giving more than one easy access to any island is 2. Since the two BABCO stores already give easy access to all islands, the minimum number of BABCO stores for all the islands to have easy access also equals 2.

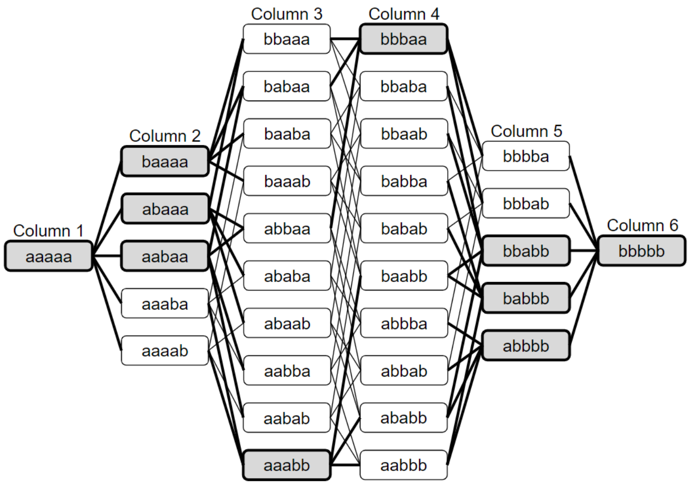
ii. → n = 4



<figure 8>

The maximum number of BABCO stores for islands in the case of n = 3 without giving more than one easy access to any island is Column 2 as shown in <figure 8>. The two stores are each located on the island in Column 1 and on the island in Column 5. The islands in Column 2 and 4 already have easy access. Therefore, it is necessary to give easy access to the islands in Column 3 in order to give easy access to all the islands in the case of n = 4. If I want to easily access all islands in Column 3 by building BABCO stores on the islands in Column 3, then there must be a BABCO store on each island in Column 3, which seems to be more than the minimum. Therefore, it is necessary to build stores on the islands in Column 2 and 4 to give easy access to the islands in Column 3. In the diagram above, if a store is built on the island baaa, then easy access can be given to the islands bbaa, baba, and baab. Also, if a store is built on the island abbb, then easy access can be given to the islands abba, abab, and aabb. The two new stores each built on the islands baaa and abbb give easy access to every island in Column 3. Therefore, the minimum number of BABCO stores to give easy access to every island for the case of n = 4 is 4.

iii. → n = 5



<figure 9>

A BABCO store on island aaaaa in Column 1 gives easy access to itself and every island in Column 2 as shown in <*figure* 9>. Each store on an island in Column 2 would give 4 islands in Column 3 easy accesses. This prevents us from building more than 3 stores in Column 2. If 3 stores are built on island baaaa, abaaa, and aabaa, a total of 9 islands out of 10 islands in Column 3 would have easy accesses leaving the island aaabb alone without any easy access. (If there are only 1 or 2 stores built-in Column 2, then more than 1 island in column 3 would lack easy access). Building a store on island aaabb in column 3 would give easy accesses to the islands in Column 2 that did not have easy accesses yet and some islands in Column 4 (aaabb, ababb, baabb). We have built a total of 5 stores so far, and there are 7 islands in column 4, all 6 islands in Column 5, and 1 island in Column 1 that do not have easy access yet. A store needs to be built on island bbbbb in Column 6 to give as many islands as possible easy accesses. This one store covers all islands in Column 5 and 6. Now, there are 7 islands in Column 4 left to cover. This can be completed by building stores on islands bbabb, babbb, and abbbb in Column 5. Therefore, the minimum number of BABCO stores to give every island easy access for the case n = 5 is 10.

Optimal solutions to parts b and c

i. → For Part (a): Maximum number of stores such that no island has more than one easy access for n = 6

There are total of 7 Columns: Column 1 contains 6C0 = 1 island, Column 2 contains 6C1 = 6 islands, Column 3 contains 6C2 = 15 islands, Column 4 contains 6C3 = 20 islands, Column 5 contains 6C4 = 15 islands, Column 6 contains 6C5 = 6 islands, and lastly Column 7 contains 6C6 = 1 island.

If there is a store on island aaaaaa in Column 1, then all of the islands in Column 2 has easy access. This prevents us from building another store in either Column 2 or Column 3 because some of the islands in Column 2 would have more than one easy access otherwise. We now have to consider how many stores can be built-in Column 4. In order not to give any island in Column 3 and in Column 5 more than one easy access, stores can be built on two islands bbbaaa and aaabbb in Column 4. These two islands have all 6 spaces for their names different, they can never have ferry route towards a single island in either Column 3 or 5. And if there is a store on island bbbbbb in Column 5, this will prevent us from building any more store in Column 6 or in Column 7. Therefore, the maximum number of BABCO stores such that no island has more than one easy access for the case n = 6 is 4. (aaaaaa, bbbaaa, aaabbb, bbbbbb)

ii. → For Part (b): The minimum number of stores that every island has at least one easy access for the n = 6.

If a store is built on the island aaaaaa in Column 1, then every island in Column 2 would have easy access. We move on to Column 2 and try to build stores while trying to give as many easy accesses as possible to the islands in Column 3. If we build stores on islands baaaaa, abaaaa, and aabaaa, then a total of 12 islands out of 15 islands in Column 3 would have easy access. This can be explained using the Root count formula i.e. Root + Root-1 + Root-2. Root count is gained as the volume of the Newton polytope shared by all polynomials in the system. We built 3 stores on the remaining 3 islands that do not have easy access yet in Column 3. We now go to right-most located island bbbbbb in column 7. We built a store on that island giving all islands in Column 6 and 7 an easy access. As in Column 3, we built another 3 stores on islands bbbabb, bbbbab, and bbbbba in Column 6 and these 3 stores would give easy accesses to 12 islands out of 15 islands in Column 4. We built 3 more stores on islands that are left alone in Column 4. The same provided easy access to every island. Therefore, the minimum number of stores that every island has at least one easy access for the case n = 6 is 14.